Two-sided polynomials for discrete control design

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In the Polynomial Toolbox, all operations with polynomials are defined using the mathematical language: the formulae are given how to manipulate with the coefficients, not speaking about what the variable z in polynomials is. However, in many applications (control theory, electrical or mechanical engineering etc.), the variable represents either the derivative operator (for continuous-time differential equations) or the shift operator (for the discrete-time recurrent equations).

In such applications, some quadratic optimization may occur. To formulate and solve it, the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t$$

or

$$(f, g) = \sum_{k=-\infty}^{\infty} f_k g_k$$

is used. In connection with it, for operator A, the operation of "conjugation" $A \to A^\star$, defined

$$(f, A^\star g) = (Af, g)$$

is useful.

With polynomials

$$A(s) = A_0 + A_1 s + \cdots + A_n s^n,$$

$$A(z) = A_0 + A_1 z + \cdots + A_n z^n,$$

where A_i are matrices of complex numbers, this leads to "conjugated transposed" polynomials

$$A^\star(s) = \overline{A'}(-s) = \overline{A'_0} - \overline{A'_1}s + \overline{A'_2}s^2 + \cdots + (-1)^n\overline{A'_n}s^n,$$

$$A^\star(z) = \overline{A'}(z^{-1}) = \overline{A_0'} + \overline{A_1'}z^{-1} + \overline{A_2'}z^{-2} + \cdots + \overline{A_n'}z^{-n}.$$

To create them, the "ctranspose" macro for polynomials has been included in Polynomial Toolbox, overloading the standard macro of this name.

For s-polynomials, this is the whole story. However, for z-polynomials, an additional modification is necessary. The resulting $A^\star(z)$ is no more polynomial in z but in z^{-1}. The algebra of polynomials

$$A(z) = A_0 + A_1 z + \cdots + A_n z^n$$

is not capable of including this phenomenon. What we need is the algebra of "two-sided polynomials"

$$A(z) = A_{-m}z^{-m} + \cdots + A_{-1}z^{-1} + A_0 + A_1 z + \cdots + A_n z^n.$$

So, in Polynomial Toolbox Version 3, in addition to class "pol", there is also class "tsp", both equipped with all arithmetic operations and mutual conversions. The user need not take care about classes; all object creations and class conversions are performed automatically. The tsp objects occur in macros for special polynomial equations, connected with quadratic optimizations, e.g. linear symmetric polynomial equation and spectral factorization.

It is worth to note, how this problems was solved in the previous Version 2. To result in z-polynomial, ctranspose operation was defined

$$A^\star = z^n\overline{A'}(z^{-1}) = \overline{A'}_n + \overline{A'}_{n-1}z + \cdots + \overline{A_0'}z^n.$$

However, in this formula, the degree n played a role. If several polynomials with different degrees occured, the user's responsibility was to keep track of the degrees during the computation, e.g. sometimes to compute the maximum of them etc. There was a risk of errors in this work.

In version 3, the user is free of this care; the system performs it automatically, the degrees being never explicitly used. This is made possible by the fact that now the ctranspose operation has the following nice properties

$$(A+B)^\star = A^\star + B^\star,$$

$$(AB)^\star = B^\star A^\star,$$

$$A^{\star\star} = A$$

whatever the degrees are.